Progress towards D=4, N=2 pure de Sitter supergravity

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July 8, 2019

Abstract: Using a general derivation framework we investigate which usual rule of supergravity should be modified to progress towards $N = 2$ pure de Sitter supergravity without having to introduce any other field than that of the graviton, the photon, and the gravitino. The derivation is in $D = 4$ dimensions, its action is locally supersymmetric up to quartic terms, and no supersymmetry breaking is required.

"One should not desist from pursuing to the end the path of the relativistic field theory."
A. Einstein

"A great deal of my work is just playing with equations and seeing what they give."
P. Dirac

The last version of this paper can be found at *http://www.sugra.org*.

1 Introduction

Supergravity is a wonderful achievement at the crossroads between the works of A. Einstein and P. Dirac [1–5]. Noteworthily, almost since its discovery forty years ago, $N = 1$ pure supergravity is known to be anti-de Sitter with a negative cosmological constant when only the graviton bosonic field e_μ^m and the gravitino fermionic field ψ_μ are considered [2]. It is only recently that $N = 1$ pure de Sitter supergravity with a positive cosmological constant has been derived by assuming the existence of an additional nilpotent Goldstino fermionic field χ associated with the spontaneous supersymmetry breaking [6].

Given the no-go theorems on the subject [7, 8], it is obvious that hoping to derive $N = 2$ pure de Sitter supergravity in a different way than [6] requires an heterodox approach. In this paper we present such an approach that consists in modifying the usual rule for counting spinors in supergravity.

For motivation to derive de Sitter supergravity we refer the reader to the Introduction of [6].

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2 A reminder on spinors and their Lorentz derivative

Given any Dirac spinor $\chi \equiv (\chi)_\alpha$ whose components are four complex Grassmann-odd variables, we follow Sec. 3.2 of [5] to define its Dirac conjugate $\bar{\chi} \equiv (\chi)^\alpha$ and its charge conjugate $\chi^C \equiv (\chi^C)_\alpha$. That is[1]

$$\bar{\chi} \equiv i\chi^\dagger\gamma^0 \,, \tag{1}$$
$$\chi^C \equiv i\gamma^0 C^\dagger \chi^* \,, \tag{2}$$

where C is the charge conjugation matrix having the properties

$$C^{-1} = C^\dagger \,,\, C^T = -C \Longrightarrow CC^* = C^*C = -I \,, \tag{3}$$
$$(C\gamma_*)^T = -C\gamma_* \,. \tag{4}$$
$$(C\gamma^m)^T = C\gamma^m \,, \tag{5}$$
$$(C\gamma^{mn})^T = C\gamma^{mn} \,, \tag{6}$$
$$(C\gamma^{mnr})^T = -C\gamma^{mnr} \,, \tag{7}$$
$$(C\gamma^{mnrs})^T = -C\gamma^{mnrs} \,. \tag{8}$$

From the definitions (1),(2) and the properties (3),(5), one can see that

$$\overline{\chi^C} \equiv i(\chi^C)^\dagger\gamma^0 = \chi^T C \,, \tag{9}$$

which is the expression for the so-called Majorana conjugate of χ.

Supergravity uses Majorana spinors that are defined by the property

$$\chi^C = \chi \,. \tag{10}$$

From (1),(9) one can see that the Dirac conjugate of a Majorana spinor is equal to its Majorana conjugate[2]

$$\chi^C = \chi \Longrightarrow \bar{\chi} = i\chi^\dagger\gamma^0 = \chi^T C \,. \tag{11}$$

As usual the Lorentz derivative of a spinor is defined by[3]

$$D_\mu\chi \equiv \partial_\mu\chi + \frac{1}{4}\omega_{\mu mn}\gamma^{mn}\chi \Longrightarrow D_\mu\bar{\chi} = \partial_\mu\bar{\chi} - \frac{1}{4}\omega_{\mu mn}\bar{\chi}\gamma^{mn} \,, \tag{12}$$

where $\omega_{\mu mn}$ is the so-called spin connection.

[1] Our conventions are those of [5]: the metric signature is $(- + ++)$; the constant γ-matrices are defined by $\gamma^m\gamma^n + \gamma^n\gamma^m = 2\eta^{mn}I$ where I is the unit matrix $\Rightarrow (\gamma^0)^2 = -I$ and $(\gamma^k)^2 = I$ with $k = 1, 2, 3$; $(\gamma^m)^\dagger = \gamma^0\gamma^m\gamma^0 \Rightarrow (\gamma^0)^\dagger = -\gamma^0$, $(\gamma^k)^\dagger = \gamma^k$; $\gamma_m \equiv \eta_{mn}\gamma^n \Rightarrow \gamma_0 = -\gamma^0$ and $\gamma_k = \gamma^k$; $\gamma_* \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma_0\gamma_1\gamma_2\gamma_3 \Rightarrow (\gamma_*)^2 = I$, $(\gamma_*)^\dagger = \gamma_*$ and $\gamma_*\gamma^m = -\gamma^m\gamma_*$; $\gamma^{mn} \equiv \gamma^{[m}\gamma^{n]} = \frac{1}{2}(\gamma^m\gamma^n - \gamma^n\gamma^m)$ antisymmetric with strength one, and so on for γ^{mnr} and γ^{mnrs} ; $\epsilon^{\mu\nu\rho\sigma} \equiv e\,\epsilon^{mnrs}e^\mu_m e^\nu_n e^\rho_r e^\sigma_s \Rightarrow \epsilon_{\mu\nu\rho\sigma} = e^{-1}\epsilon_{mnrs}e^m_\mu e^n_\nu e^r_\rho e^s_\sigma$ with $\epsilon_{0123} = +1 = -\epsilon^{0123}$.

[2] In the rest of the paper we shall use the Majorana conjugate to simplify the calculations since it does not involve complex conjugation.

[3] For a Majorana spinor the expression for $D_\mu\bar{\chi}$ follows directly from (11) and (3),(6): it is easy to verify that $\overline{(D_\mu\chi)} = (D_\mu\chi)^T C = D_\mu\bar{\chi}$.

3 The derivation framework up to quartic terms

In this section we present the general framework to derive $N = 2$ pure (anti-)de Sitter supergravity up to quartic (four-fermion) terms. To emphasize that the derivation is up to quartic terms we will systematically multiply quadratic terms by the constant λ and consider that $\lambda^2 \simeq 0$ in the calculations.

The derivation framework depends on seven real constant matrices $S_1, S_2, A_1, A_2, M_1, M_2, M_3$ that will be specified in the next sections. In this section it is only supposed that the two matrices S are symmetric ($S = S^T$) and the two matrices A are antisymmetric ($A = -A^T$).

The action considered is similar to the one of $N = 2$ pure anti-de Sitter supergravity (see Sec. 2.8 of [4] and references therein). Therefore, the action shall be based on the graviton real bosonic field e_μ^m, the cosmological real constant Λ, the gravitino complex fermionic field $\psi_\mu^i \equiv (\psi_\mu^i)_\alpha$ and the photon real bosonic field A_μ with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ as usual. Therefore also, the action shall be based on spinor-ansatz bilinears[4] $M^{ij}(\bar{\chi}^i N \xi^j)$ where χ^i and ξ^i are any two Majonara spinor-ansatzes, N is any matrix obtained from products of γ-matrices, and M is any real constant matrix.

The action is the sum of the Einstein-Hilbert term, the cosmological constant term, the Rarita-Schwinger term, two gauged terms, the Pauli term, and quartic terms[5]:

$$S \;=\; S_{\text{EH}} + S_\Lambda + S_M + S_{\text{RS}} + S_{\text{g}\Lambda} + S_{\text{gM}} + S_{\text{P}} + \lambda^2(\text{quartic terms})\,, \tag{13}$$

where[6]

$$S_{\text{EH}} \;=\; \int dx^4\, e\, R \equiv \int dx^4\, e\, e_m^\mu e_n^\nu R_{\mu\nu}{}^{mn} = -\frac{1}{4}\int dx^4 \epsilon^{\mu\nu\rho\sigma}\epsilon_{mnrs}R_{\mu\nu}{}^{mn}e_\rho^r e_\sigma^s\,, \tag{14}$$

$$S_\Lambda \;=\; \pm 2\int dx^4\, e\,\Lambda\,, \tag{15}$$

$$S_M \;=\; -\frac{1}{4}\int dx^4\, e\, F_{\mu\nu}F^{\mu\nu}\,, \tag{16}$$

$$S_{\text{RS}} \;=\; -\frac{1}{2}\int dx^4\, e\, S_1^{ij}\lambda(\bar{\psi}_\mu^i\gamma^{\mu\nu\rho}D_\nu\psi_\rho^j) = \frac{i}{2}\int dx^4\epsilon^{\mu\nu\rho\sigma}S_1^{ij}\lambda(\bar{\psi}_\mu^i\gamma_*\gamma_\nu D_\rho\psi_\sigma^j)\,, \tag{17}$$

$$S_{\text{g}\Lambda} \;=\; \frac{1}{2}\sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, S_2^{ij}\lambda(\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j)\,, \tag{18}$$

$$S_{\text{gM}} \;=\; \frac{1}{4}\sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, A_\mu A_1^{ij}\lambda(\bar{\psi}_\nu^i\gamma^{\mu\nu\rho}\psi_\rho^j) = \frac{i}{4}\sqrt{\frac{\Lambda}{3}}\int dx^4\epsilon^{\mu\nu\rho\sigma}A_\mu A_1^{ij}\lambda(\bar{\psi}_\nu^i\gamma_*\gamma_\rho\psi_\sigma^j)\,. \tag{19}$$

$$S_{\text{P}} \;=\; \frac{1}{8}\int dx^4\, e\, A_2^{ij}\lambda(\bar{\psi}_\mu^i\gamma^{[\mu}\gamma_{\rho\sigma}\gamma^{\nu]}\psi_\nu^j)F^{\rho\sigma}\,, \tag{20}$$

with

$$R_{\mu\nu}{}^{mn} \;\equiv\; \partial_\mu\omega_\nu{}^{mn} - \partial_\nu\omega_\mu{}^{mn} + \omega_\mu{}^m{}_r\,\omega_\nu{}^{rn} - \omega_\nu{}^m{}_r\,\omega_\mu{}^{rn}\,, \tag{21}$$

$$D_\mu\psi_\nu^i \;\equiv\; \partial_\mu\psi_\nu^i + \frac{1}{4}\omega_{\mu mn}\gamma^{mn}\psi_\nu^i \implies D_\mu\bar{\psi}_\nu^i = \partial_\mu\bar{\psi}_\nu^i - \frac{1}{4}\omega_{\mu mn}\bar{\psi}_\nu^i\gamma^{mn}\,. \tag{22}$$

[4]In order for the action to be real, it is proven in Appendix A that such spinor-ansatz bilinears are real.

[5]As in [4] we use the convention that the gravitational constant κ^2 is normalized by $2\kappa^2 \equiv 16\pi G c^{-4} = 1$.

[6]As usual in supergravity: $e = \det e_\mu^m$; $\epsilon_{\mu\nu\rho\sigma} \equiv e^{-1}\epsilon_{mnrs}e_\mu^m e_\nu^n e_\rho^r e_\sigma^s$, $\epsilon^{\mu\nu\rho\sigma} \equiv e\,\epsilon^{mnrs}e_m^\mu e_n^\nu e_r^\rho e_s^\sigma$ with $\epsilon_{0123} = +1 = -\epsilon^{0123}$ and $\epsilon^{\mu\nu\rho\sigma}\epsilon_{mnrs}e_\rho^r e_\sigma^s = -2\,e\,(e_m^\mu e_n^\nu - e_n^\mu e_m^\nu)$; $\gamma^{\mu\nu\rho} \equiv \gamma^{mnr}e_m^\mu e_n^\nu e_r^\rho = -e^{-1}\epsilon^{\mu\nu\rho\sigma}i\gamma_*\gamma_\sigma$ with $\gamma_\sigma \equiv e_\sigma^m\gamma_m$.

The plus and minus signs in the cosmological constant term (15) correspond respectively to anti-de Sitter supergravity and de Sitter supergravity.

As explained in Sec. 4 of [7], it is important that the relative sign between the graviton term (14) and the photon term (16) is negative in order to avoid ghosts[7]*.*

Using the so-called 1.5 order formalism, our goal is to show that the action (13) is invariant $\delta S = 0$ up to quartic terms under the following local supersymmetry transformations involving the spinor-ansatz $\epsilon^i \equiv (\epsilon^i)_\alpha$ as supersymmetry parameter[8]:

$$\delta e^m_\mu \;=\; \frac{1}{4} S_1^{ij} \lambda(\bar\epsilon^i \gamma^m \psi^j_\mu) \implies \delta e^\mu_m = -\frac{1}{4} S_1^{ij} \lambda(\bar\epsilon^i \gamma^\mu \psi^j_m) \implies \delta e = \frac{1}{4}\, e\, S_1^{ij} \lambda(\bar\epsilon^i \gamma^\rho \psi^j_\rho)\,, \tag{23}$$

$$\delta A_\mu \;=\; \frac{1}{2} A_2^{ij} \lambda(\bar\epsilon^i \psi^j_\mu)\,, \tag{24}$$

$$\delta \psi^i_\mu \;=\; D_\mu \epsilon^i + \frac{1}{2}\sqrt{\frac{\Lambda}{3}} M_1^{ik} \gamma_\mu \epsilon^k + \frac{1}{2}\sqrt{\frac{\Lambda}{3}} A_\mu M_2^{ik} \epsilon^k - \frac{1}{8} M_3^{ik} \gamma^{\alpha\beta} \gamma_\mu \epsilon^k F_{\alpha\beta}\,, \tag{25}$$

$$\implies \delta \bar\psi^i_\mu = D_\mu \bar\epsilon^i - \frac{1}{2}\sqrt{\frac{\Lambda}{3}} M_1^{ik} \bar\epsilon^k \gamma_\mu + \frac{1}{2}\sqrt{\frac{\Lambda}{3}} A_\mu M_2^{ik} \bar\epsilon^k - \frac{1}{8} M_3^{ik} \bar\epsilon^k \gamma_\mu \gamma^{\alpha\beta} F_{\alpha\beta}\,. \tag{26}$$

As stated in [3] the 1.5 order formalism is nothing else than the Palatini trick of general relativity extended to supergravity. We shall treat the spin connection ${\omega_\mu}^{mn} = -{\omega_\mu}^{nm}$ as an independent field and impose that the variation of the action (13) with respect to it vanishes:

$$\delta_\omega S \;=\; \delta_\omega S_{\rm EH} + \delta_\omega S_{\rm RS} = 0\,, \tag{27}$$

where[9]

$$\delta_\omega S_{\rm EH} \;=\; \int dx^4\, \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs}\, \delta{\omega_\mu}^{mn} e^r_\nu\, D_\rho e^s_\sigma\,, \tag{28}$$

$$\delta_\omega S_{\rm RS} \;=\; -\frac{1}{8}\int dx^4\, \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnrs}\, \delta{\omega_\mu}^{mn}\, e^r_\nu\, S_1^{ij} \lambda(\bar\psi^i_\rho \gamma^s \psi^j_\sigma)\,, \tag{29}$$

which leads to the equation

$$D_{[\rho} e^s_{\sigma]} \;=\; \frac{1}{8} S_1^{ij} \lambda(\bar\psi^i_{[\rho} \gamma^s \psi^j_{\sigma]})\,. \tag{30}$$

This equation for ${\omega_\mu}^{mn}$ can be solved to find

$${\omega_\mu}^{mn} = {\omega_\mu}^{mn}(e) + {K_\mu}^{mn}(\psi)\,, \tag{31}$$

where ${\omega_\mu}^{mn}(e)$ is the torsionless spin connection

$${\omega_\mu}^{mn}(e) = \frac{1}{2} e^{m\rho}(\partial_\mu e^n_\rho - \partial_\rho e^n_\mu) - \frac{1}{2} e^{n\rho}(\partial_\mu e^m_\rho - \partial_\rho e^m_\mu) - \frac{1}{2} e^{m\rho} e^{n\sigma}(\partial_\rho e^r_\sigma - \partial_\sigma e^r_\rho) e_{r\mu}\,, \tag{32}$$

and ${K_\mu}^{mn}(\psi)$ is the so-called contortion tensor given by

$${K_\mu}^{mn}(\psi) = \frac{1}{8} S_1^{ij} \lambda(\bar\psi^i_\mu \gamma^m \psi^{jn} - \bar\psi^i_\mu \gamma^n \psi^{jm} + \bar\psi^{im} \gamma_\mu \psi^{jn})\,. \tag{33}$$

[7]Note that the conventions used in [7] change the sign of the graviton term.

[8]For Majorana spinors the expression for $\delta\bar\psi^i_\mu$ follows directly from (11) and (3),(5),(6): it is easy to verify that $\overline{(\delta\psi^i_\mu)} = (\delta\psi^i_\mu)^T C = \delta\bar\psi^i_\mu$.

[9]The first variation requires partial integration.

Following the spirit of the 1.5 order formalism, the result (30) shall be taken into account in the next variations. The fact that this result is proportional to λ will help to simplify the calculations since it is supposed that $\lambda^2 \simeq 0$.

First we vary the action (13) with respect to the graviton e_μ^m up to λ^2-terms:

$$\delta_e S_{\rm EH} = -\frac{1}{8}\int dx^4\, \epsilon^{\mu\nu\rho\sigma}\epsilon_{mnrs}R_{\mu\nu}{}^{mn}e_\rho^r\, S_1^{ij}\lambda(\bar\epsilon^i\,\gamma^s\psi_\sigma^j)\,, \tag{34}$$

$$\begin{aligned}\delta_e S_M &= -\frac{1}{16}\int dx^4\; e\, S_1^{ij}\lambda(\bar\epsilon^i\,\gamma^\rho\psi_\rho^j)F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\int dx^4\; e\, S_1^{ij}\lambda(\bar\epsilon^i\,\gamma^\mu\psi_\nu^j)F_{\mu\rho}F^{\nu\rho}\,, \\ &= -\frac{1}{2}\int dx^4\, e\,(e_n^\nu R_{\mu\nu}{}^{mn} - \frac{1}{2}e_\mu^m R)\, S_1^{ij}\lambda(\bar\epsilon^i\,\gamma^\mu\psi_m^j)\,,\end{aligned} \tag{35}$$

$$\delta_e S_\Lambda = \pm\frac{\Lambda}{2}\int dx^4\, e\, S_1^{ij}\lambda(\bar\epsilon^i\,\gamma^\rho\psi_\rho^j)\,. \tag{36}$$

One can see that $\delta_e S_{\rm RS}\simeq 0$, $\delta_e S_{\rm g\Lambda}\simeq 0$, $\delta_e S_{\rm gM}\simeq 0$ and $\delta_e S_{\rm P}\simeq 0$ since it is supposed that $\lambda^2\simeq 0$.

Next we vary[10] the action (13) with respect to the gravitino-ansatz ψ_μ^i up to λ^2-terms[11]:

$$\begin{aligned}\delta_\psi S_{\rm RS} = &+\frac{1}{8}\int dx^4\, \epsilon^{\mu\nu\rho\sigma}\epsilon_{mnrs}R_{\mu\nu}{}^{mn}e_\rho^r\, S_1^{ij}\lambda(\bar\epsilon^i\gamma^s\psi_\sigma^j) + \sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, S_1^{jk}M_1^{ki}\lambda(\bar\epsilon^i\gamma^{\mu\nu}D_\mu\psi_\nu^j) \\ &-\frac{1}{2}\sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, A_\mu S_1^{jk}M_2^{ki}\lambda(\bar\epsilon^i\gamma^{\mu\nu\rho}D_\nu\psi_\rho^j) - \frac{1}{2}\int dx^4\, e\, S_1^{jk}M_3^{ki}\lambda(\bar\epsilon^i D_\mu\psi_\nu^j)F^{\mu\nu} \\ &-\frac{1}{4}\int dx^4\, e\, S_1^{jk}M_3^{ki}\lambda(\bar\epsilon^i\gamma^{\mu\nu\rho\sigma}D_\mu\psi_\nu^j)F_{\rho\sigma}\,,\end{aligned} \tag{37}$$

$$\begin{aligned}\delta_\psi S_{\rm g\Lambda} = &-\sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, S_2^{ij}\lambda(\bar\epsilon^i\gamma^{\mu\nu}D_\mu\psi_\nu^j) - \frac{\Lambda}{2}\int dx^4\, e\, S_2^{jk}M_1^{ki}\lambda(\bar\epsilon^i\gamma^\rho\psi_\rho^j) \\ &+\frac{\Lambda}{6}\int dx^4\, e\, A_\mu S_2^{jk}M_2^{ki}\lambda(\bar\epsilon^i\gamma^{\mu\nu}\psi_\nu^j) - \frac{1}{4}\sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, S_2^{jk}M_3^{ki}\lambda(\bar\epsilon^i\gamma_\mu\psi_\nu^j)F^{\mu\nu} \\ &+\frac{1}{8}\sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, S_2^{jk}M_3^{ki}\lambda(\bar\epsilon^i\gamma^{\mu\nu\rho}\psi_\mu^j)F_{\nu\rho}\,,\end{aligned} \tag{38}$$

$$\begin{aligned}\delta_\psi S_{\rm gM} = &+\frac{1}{4}\sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, A_1^{ij}\lambda(\bar\epsilon^i\gamma^{\mu\nu\rho}\psi_\mu^j)F_{\nu\rho} - \frac{1}{2}\sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, A_\mu A_1^{ij}\lambda(\bar\epsilon^i\gamma^{\mu\nu\rho}D_\nu\psi_\rho^j) \\ &-\frac{\Lambda}{6}\int dx^4\, e\, A_\mu A_1^{jk}M_1^{ki}\lambda(\bar\epsilon^i\gamma^{\mu\nu}\psi_\nu^j) + \frac{1}{4}\sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, A_\mu A_1^{jk}M_3^{ki}\lambda(\bar\epsilon^i\psi_\nu^j)F^{\mu\nu} \\ &+\frac{1}{8}\sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, A_\mu A_1^{jk}M_3^{ki}\lambda(\bar\epsilon^i\gamma^{\mu\nu\rho\sigma}\psi_\nu^j)F_{\rho\sigma}\,,\end{aligned} \tag{39}$$

$$\begin{aligned}\delta_\psi S_{\rm P} = &+\frac{1}{2}\int dx^4 A_2^{ij}\lambda(\bar\epsilon^i\psi_\mu^j)D_\nu(e\,F^{\mu\nu}) - \frac{1}{2}\int dx^4\, e\, A_2^{ij}\lambda(\bar\epsilon^i D_\mu\psi_\nu^j)F^{\mu\nu} \\ &-\frac{1}{4}\int dx^4\, e\, A_2^{ij}\lambda(\bar\epsilon^i\gamma^{\mu\nu\rho\sigma}D_\mu\psi_\nu^j)F_{\rho\sigma} + \frac{1}{4}\sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, A_2^{jk}M_1^{ki}\lambda(\bar\epsilon^i\gamma_\mu\psi_\nu^j)F^{\mu\nu}\end{aligned}$$

[10]The calculations require spinor-flips that are performed by taking the transpose of the spinor-anzatz bilinears. As usual, these spinor-flips are performed by taking into account the properties (3) to (8) and by incorporating a minus sign obtained by changing the order of the Grassmann-odd valued spinor components.

[11]These variations require partial integration.

$$+\frac{1}{8}\sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, A_2^{jk}M_1^{ki}\lambda(\bar{\epsilon}^i\gamma^{\mu\nu\rho}\psi_\mu^j)F_{\nu\rho}-\frac{1}{4}\sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, A_\mu A_2^{jk}M_2^{ki}\lambda(\bar{\epsilon}^i\psi_\nu^j)F^{\mu\nu}$$
$$-\frac{1}{8}\sqrt{\frac{\Lambda}{3}}\int dx^4\, e\, A_\mu A_2^{jk}M_2^{ki}\lambda(\bar{\epsilon}^i\gamma^{\mu\nu\rho\sigma}\psi_\nu^j)F_{\rho\sigma}-\frac{1}{16}\int dx^4\, e\, A_2^{jk}M_3^{ki}\lambda(\bar{\epsilon}^i\,\gamma^\rho\psi_\rho^j)F_{\mu\nu}F^{\mu\nu}$$
$$+\frac{1}{4}\int dx^4\;\, e\, A_2^{jk}M_3^{ki}\lambda(\bar{\epsilon}^i\,\gamma^\mu\psi_\nu^j)F_{\mu\rho}F^{\nu\rho}\,. \tag{40}$$

Finally we vary the action (13) with respect to the photon A_μ up to λ^2-terms[12]:

$$\delta_A S_M \;=\; -\frac{1}{2}\int dx^4 A_2^{ij}\lambda(\bar{\epsilon}^i\psi_\mu^j)D_\nu(e\,F^{\mu\nu})\,. \tag{41}$$

One can see that $\delta_A S_{\rm gM}\simeq 0$ and $\delta_A S_{\rm P}\simeq 0$ since it is supposed that $\lambda^2\simeq 0$.

The action (13) is invariant $\delta S=0$ up to quartic terms if the sum of the expressions (34) to (41) vanishes, which is the case when the seven real constant matrices satisfy the conditions

$$\begin{aligned} M_1^2 &= \pm I\,,\\ S_1M_2 &= A_1\,,\\ S_1M_3 &= A_2\,,\\ S_2M_2 &= A_1M_1\,,\\ S_2M_3 &= A_1\,,\\ A_1M_3 &= A_2M_2\,,\\ A_2M_3 &= -S_1\,, \end{aligned} \tag{42}$$

where I is the identity matrix.

4 D=4, N=2 pure anti-de Sitter supergravity up to quartic terms

In this section we consider the plus sign in the first condition of (42).

The following 2 x 2 matrices satisfy the conditions (42) when it is a plus sign in the first condition:

$$S_1=S_2=M_1=I\;,\;A_1=A_2=M_2=M_3=E\,, \tag{43}$$

where

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix}\;\;,\;\;E=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\,. \tag{44}$$

The solution (43) is not the only one. The following 2 x 2 matrices also satisfy the conditions (42) when it is a plus sign in the first condition:

$$S_1=S_2=-I\;,\;M_1=I\;,\;A_1=A_2=-E\;,\;M_2=M_3=E\,. \tag{45}$$

[12]The variation requires partial integration.

It is easy to verify that the solution (45) is equivalent to (43) by applying the transformations

$$\begin{aligned} \psi^1_\mu \longrightarrow i\psi^1_\mu \ , \ \psi^2_\mu \longrightarrow i\psi^2_\mu \, , \\ \bar{\psi}^1_\mu \longrightarrow i\bar{\psi}^1_\mu \ , \ \bar{\psi}^2_\mu \longrightarrow i\bar{\psi}^2_\mu \, , \\ \epsilon^1 \longrightarrow i\epsilon^1 \ , \ \epsilon^2 \longrightarrow i\epsilon^2 \, , \\ \bar{\epsilon}^1 \longrightarrow i\bar{\epsilon}^1 \ , \ \bar{\epsilon}^2 \longrightarrow i\bar{\epsilon}^2 \, , \end{aligned} \tag{46}$$

to all tensorial expressions.

It is also possible to combine the solutions (43) and (45) into a more complicated solution. The following 4 x 4 matrices[13] also satisfy the conditions (42) when it is a plus sign in the first condition:

$$S_1 = S_2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \ A_1 = S_1 M_2 = A_2 = S_1 M_3 = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix},$$
$$M_1 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \ M_2 = M_3 = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}. \tag{47}$$

It is easy to verify that the solution (47) is equivalent to (43) by applying the transformations

$$\begin{aligned} \psi^1_\mu \longrightarrow \sqrt{2}\psi^1_\mu \ , \ \psi^2_\mu \longrightarrow \sqrt{2}\psi^2_\mu \ , \ \psi^3_\mu \longrightarrow i\sqrt{2}\psi^1_\mu \ , \ \psi^4_\mu \longrightarrow i\sqrt{2}\psi^2_\mu \, , \\ \bar{\psi}^1_\mu \longrightarrow \sqrt{2}\bar{\psi}^1_\mu \ , \ \bar{\psi}^2_\mu \longrightarrow \sqrt{2}\bar{\psi}^2_\mu \ , \ \bar{\psi}^3_\mu \longrightarrow i\sqrt{2}\bar{\psi}^1_\mu \ , \ \bar{\psi}^4_\mu \longrightarrow i\sqrt{2}\bar{\psi}^2_\mu \, , \\ \epsilon^1 \longrightarrow \sqrt{2}\epsilon^1 \ , \ \epsilon^2 \longrightarrow \sqrt{2}\epsilon^2 \ , \ \epsilon^3 \longrightarrow i\sqrt{2}\epsilon^1 \ , \ \epsilon^4 \longrightarrow i\sqrt{2}\epsilon^2 \, , \\ \bar{\epsilon}^1 \longrightarrow \sqrt{2}\bar{\epsilon}^1 \ , \ \bar{\epsilon}^2 \longrightarrow \sqrt{2}\bar{\epsilon}^2 \ , \ \bar{\epsilon}^3 \longrightarrow i\sqrt{2}\bar{\epsilon}^1 \ , \ \bar{\epsilon}^4 \longrightarrow i\sqrt{2}\bar{\epsilon}^2 \, , \end{aligned} \tag{48}$$

to all tensorial expressions.

The solution (47) is pedantic in the context of this section because the transformations (48) gives back the usual solution (43). In the next section we shall consider a solution similar to (47) for which the transformations (48) are not allowed.

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In this section we consider the minus sign in the first condition of (42).

One can verify that there are no 2 x 2 or 3 x 3 real constant matrices satisfying the conditions (42) when it is a minus sign in the first condition. However, these conditions are satisfied by the following 4 x 4 matrices:

$$S_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \ S_2 = S_1 M_1 = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \ A_1 = S_1 M_2 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \ A_2 = S_1 M_3 = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix},$$
$$M_1 = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}, \ M_2 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \ M_3 = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \tag{49}$$

where I and E are given by (44).

Using the usual rule for counting spinors in supergravity, it seems that the solution (49) would be part of $D = 4, N = 4$ pure de Sitter supergravity. However, this cannot be the case

[13] In this case the index i ranges from 1 to 4, which means that all tensorial expressions involve four gravitinos ψ^i_μ and four supersymmetry parameters ϵ^i.

because $D = 4, N = 4$ pure supergravity [9] is characterized, as all usual extended supergravities, by having $S_1 = I$ where I is the identity matrix. The following transformations applied to (49)

$$\begin{aligned}
&\psi^1_\mu \longrightarrow \psi^1_\mu \ , \ \psi^2_\mu \longrightarrow \psi^2_\mu \ , \ \psi^3_\mu \longrightarrow i\psi^3_\mu \ , \ \psi^4_\mu \longrightarrow i\psi^4_\mu \, , \\
&\bar\psi^1_\mu \longrightarrow \bar\psi^1_\mu \ , \ \bar\psi^2_\mu \longrightarrow \bar\psi^2_\mu \ , \ \bar\psi^3_\mu \longrightarrow i\bar\psi^3_\mu \ , \ \bar\psi^4_\mu \longrightarrow i\bar\psi^4_\mu \, , \\
&\epsilon^1 \longrightarrow \epsilon^1 \ , \ \epsilon^2 \longrightarrow \epsilon^2 \ , \ \epsilon^3 \longrightarrow i\epsilon^3 \ , \ \epsilon^4 \longrightarrow i\epsilon^4 \, , \\
&\bar\epsilon^1 \longrightarrow \bar\epsilon^1 \ , \ \bar\epsilon^2 \longrightarrow \bar\epsilon^2 \ , \ \bar\epsilon^3 \longrightarrow i\bar\epsilon^3 \ , \ \bar\epsilon^4 \longrightarrow i\bar\epsilon^4 \, ,
\end{aligned} \tag{50}$$

would lead to $S_1 = I$ but also to imaginary matrices for the off-diagonal matrices S_2, A_1, M_1, M_2 and therefore to an infamous complex action (13). For the same reason, one can see that the transformations (48) may not be applied to the solution (49).

We therefore propose to modify the usual rule for counting spinors in supergravity by doubling their usual number and consider (49) as the solution for progressing towards $D = 4, N = 2$ pure de sitter supergravity.

6 Discussion

6.1 D=4, N=2 pure supergravity up to quartic terms

Setting $S_2 = A_1 = M_1 = M_2 = 0$ in (49) allows to apply the transformations (48), which leads to D=4, N=2 pure supergravity (see Sec. 2.7 of [4] and references therein) up to quartic terms.

6.2 Killing spinors

The Killing spinor analysis is similar to the one given in Sec. 2.2.3 of [9]. When $\psi^i_\mu = 0$ and $A_\mu = 0$ the field equations derived from the action (13) are $e^\nu_n R_{\mu\nu}{}^{mn} - \frac{1}{2} e^m_\mu R + e^m_\mu \Lambda = 0$ whose homogeneous solution is de Sitter space with curvature tensor

$$R_{\mu\nu}{}^{mn} \ = \ \frac{\Lambda}{3}(e^m_\mu e^n_\nu - e^m_\nu e^n_\mu)\, . \tag{51}$$

The conditions $\psi^i_\mu = 0 \Rightarrow \delta\psi^i_\mu = 0$ and $A_\mu = 0$ lead from (25) to the Killing spinor equation $\hat D_\mu \epsilon^i \equiv D_\mu \epsilon^i + \frac{1}{2}\sqrt{\frac{\Lambda}{3}} M_1^{ij} \gamma_\mu \epsilon^j = 0$ whose integrability condition is[14]

$$[\hat D_\mu, \hat D_\nu]\epsilon^i \ = \ [\frac{1}{4} R_{\mu\nu}{}^{mn}\gamma_{mn} - \frac{\Lambda}{6}\gamma_{\mu\nu}]\epsilon^i = 0\, . \tag{52}$$

Substituing (51) into (52), one can see that the integrability condition is obeyed for any supersymmetry parameter-ansatz ϵ^i.

7 Conclusion

We emphasize that we have "only" derived $D = 4, N = 2$ pure de Sitter supergravity up to quartic terms. The good point is that its action (13) has the right relative sign between the graviton and the photon terms. The bad point is that one has to modify the usual rule for counting spinors in supergravity by doubling their usual number.

[14] The calculation takes into account $(M_1)^2 = -I$ from (42) and $D_{[\rho} e^s_{\sigma]} = 0$ obtained from (30) when $\psi^i_\mu = 0$.

Acknowledgments

I am grateful to A. Van Proeyen for valuable comments and stimulating suggestions.

I also would like to thank P. van Nieuwenhuizen for short but enlightening mail exchanges.

A Proof that spinor-bilinears are real

In this appendix we follow Sec. 3.2.4 of [5]. Let's consider the spinor-bilinear $\bar{\chi}N\xi$ which is based on any two Majorana spinors χ and ξ and where N is any matrix obtained from products of γ-matrices. For a Majorana spinor χ it is easy to show from (1),(2),(11) that $\bar{\chi} = \chi^T C$, $\chi = B^{-1}\chi^*$ where $B = iC\gamma^0 \Rightarrow B^{-1} = i\gamma^0 C^\dagger$. For a Majorana spinor χ it is also easy to show with (3),(5),(9) that $\bar{\chi} = -(\bar{\chi})^* B$. From (3)-(8) and footnote 1, it can also be shown that $B^{-1}N^*B = N$. Therefore, we have[15]

$$(\bar{\chi}N\xi)^* \;=\; -(\bar{\chi})^* N^* \xi^* = -(\bar{\chi})^* B B^{-1} N^* B B^{-1} \xi^* = \bar{\chi}N\xi \,,$$

which proves that the spinor-bilinear $\bar{\chi}N\xi$ is real. Therefore, one can see that the spinor-ansatz bilinear $M^{ij}(\bar{\chi}^i N \xi^j)$ is real for any real constant matrix M.

[15] As in Sec. 3.2.4 of [5] we use the convention that the order of Grassmann-odd numbers is reversed in the process of complex conjugation: $(\chi\xi)^* = \xi^*\chi^* = -\chi^*\xi^*$.

References

[1] D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, Phys. Rev. D **13**, 3214 (1976);
S. Deser and B. Zumino, Phys. Lett. B **62**, 335 (1976);
D. Z. Freedman and P. van Nieuwenhuizen, Phys. Rev. D **14**, 912 (1976).

[2] P. K. Townsend, "Cosmological Constant in Supergravity," Phys. Rev. D **15**, 2802 (1977).

[3] P. van Nieuwenhuizen, A. Rebhan, D. V. Vassilevich, and R. Wimmer, "Boundary Terms in Supergravity and Supersymmetry," Int. J. Mod. Phys. **15**, 1643 (2006). [arXiv:hep-th/0606075]

[4] For a pedagogical introduction to supergravity see Y. Tanii, "Introduction to Supergravity," (Springer Briefs in Math. Phys.) 130 p. (2014).

[5] For an authoritative account of supergravity see D. Z. Freedman and A. Van Proeyen, "Supergravity," (Cambridge Univ. Pr., Cambridge, UK) 607 p. (2012).

[6] E. A. Bergshoeff, D. Z. Freedman, R. Kallosh, and A. Van Proeyen, "Pure de Sitter Supergravity," Phys. Rev. D **92**, no. 8 085040 (2015). [arXiv:1507.08264 [hep-th]]

[7] K. Pilch, P. van Nieuwenhuizen and M. F. Sohnius, "De Sitter superalgebras and supergravity," Commun. Math. Phys. **98**, 105 (1985).

[8] J. Lukierski and A. Nowicki, "All possible de Sitter superalgebras and the presence of ghosts," Phys. Lett. B **151**, 382 (1985).

[9] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rep. **323**, 183 (2000). [arXiv:hep-th/9905111]

[10] E. Cremmer, J. Scherk and S. Ferrara, Phys. Lett. B **74**, 61 (1978);
A. Das, Phys. Rev. D **15**, 2805 (1977).